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The exact solution and the finite size behaviour of the $Osp(1|2)$ invariant spin chain

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Abstract

We have solved exactly the $Osp(1|2)$ spin chain by the Bethe ansatz approach. Our solution is based on an equivalence between the $Osp(1|2)$ chain and certain special limit of the Izergin-Korepin vertex model. The completeness of the Bethe ansatz equations is discussed for a system with four sites and it is noted the appearance of special string structures. The Bethe ansatz presents an important phase-factor which distinguishes the even and odd sectors of the theory. The finite size properties are governed by a conformal field theory with central charge $c = 1$.

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1 Introduction

Exactly integrable vertex models possessing both fermionic and bosonic degrees of freedom provide interesting examples of interacting systems presenting a very rich structure in their spectrum. The exact solution of Hamiltonians which appears as graded permutations of bosons and fermions was first considered by Sutherland [1]. In general, the associated Boltzmann weights satisfy a graded version [2] of the Yang-Baxter equation and they have been investigated as invariants under the superalgebras $Sl(n|m)$ and $Osp(n|2m)$ [3, 4]. In particular, the basis of the quantum inverse scattering method for certain super-orthosymplectic magnets has been developed by Kulish [3]. An important example is the Perk-Schultz system [5, 6, 7], which recently has been recognized to appear on the solution of several models of correlated electrons on a lattice [8, 9]. For instance, the solution of the one dimensional supersymmetric $t-J$ model [10] is related to the Bethe ansatz properties of the $Sl(1|2)$ invariant Perk-Schultz like model [5, 8, 11]. In this sense, it seems quite important to search for solutions of other integrable systems possessing bosonic/fermionic degrees of freedom.

In this paper, we focus on the exact solution of the simplest super-orthosymplectic invariant spin chain. Their Boltzmann weight has three states per bond, one bosonic and two fermionic, and the associated spin magnet is invariant under $Osp(1|2)$ symmetry [3]. Its corresponding R-matrix [3] is given by (see also [4, 18, 19])

$$R(\lambda, \eta)_{i,i+1} = \lambda I_{i,i+1} + \eta P^g_{i,i+1} + \frac{\eta\lambda}{3\eta/2 - \lambda} E^g_{i,i+1} \tag{1}$$

where λ is the spectral variable, η is the quasi-classical parameter and $I_{i,i+1}$ is the 9×9 identity matrix. The graded permutation operator P^g has the elements $(P^g_{i,i+1})^{cd}_{ab} = (-1)^{p(a)p(b)}\delta_{a,d}\delta_{b,c}$ where $p(1) = 0$ (boson), $p(2) = p(3) = 1$ (fermions) in the order of BFF grading [8]. $(E^g_{i,i+1})^{cd}_{ab} = \alpha_{ab}\alpha^{st}_{cd}$ is the $Osp(1|2)$ Temperely-Lieb operator [18, 19] and the symbol st indicates the supertranspose operation. The matrix α on the specific BFF

grading has the following form

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{2}$$

The R-matrix (1) has the important property of being proportional to the graded operator P^g at the special point $\lambda = 0$. As a consequence, the local Hamiltonian is obtained as a logarithmic derivative of the Transfer matrix at point $\lambda = 0$ [8]. The associated Hamiltonian is then given by

$$H = -J \sum_{i=1}^{L} [P^g_{i,i+1} + \frac{2}{3} E^g_{i,i+1}] \tag{3}$$

where periodic boundary condition is implicitly assumed. The antiferromagnetic regime of (3) corresponds to $J > 0$. The $Osp(1|2)$ invariance of this system is shown in Appendix A.

In this paper we use the analytical/algebraic Bethe ansatz [14, 15] in order to compute the eigenvalues of the Hamiltonian (3). By means of a canonical transformation we are able to write (1) as a certain limit of the vertex operator of the Izergin-Korepin (IK) model [12]. The associated Bethe ansatz equation has a peculiar phase behaviour, which is important for the correct characterization of the critical properties.

This paper is organized as follows[1]. In section 2, we present the Bethe ansatz equations associated to the diagonalization of the Hamiltonian (3). In section (3), we discuss the completeness of the Bethe ansatz roots for a lattice of size $L = 4$. In particular we present evidence that special string structures may appear in the spectrum. The thermodynamic limit and the critical behaviour is computed in section 4. In section 5 we summarize our conclusions and discuss some remaining questions. In Appendix A and B we show the $Osp(1|2)$ invariance of the Hamiltonian (3) as well as its relation to the IK model, respectively. In Appendix C some results for twisted boundary conditions are discussed.

[1] A brief account of our results has appeared in ref. [17]

2 The Bethe ansatz solution

In this section we are going to argue that the problem of diagonalization of the Hamiltonian (3) is similar to that performed in the IK vertex model [14, 15]. We recall that more general treatments of the IK vertex model based on the A_2^2 algebra and its RSOS reductions can be found, for instance, in refs. [13, 16]. In order to see this equivalence, it is convenient to work with a vertex operator $\mathcal{L}(\lambda, \eta)$ satisfying the usual Yang-Baxter equation. As it has been first discussed in ref. [2], this can be done (without changing the original problem) because the R-matrix (1) is shown to be obtained [18] from a null-parity (Grassmann) braid operator. Their relation is rather simple [2]

$$\mathcal{L}_{ab}^{cd}(\lambda, \eta) = (-1)^{p(a)p(b)} R_{ab}^{cd}(\lambda, \eta) \tag{4}$$

The next step corresponds to making a redefinition of the grading to FBF, and afterwords to rewrite this operator in terms of the $SU(2)$ spin-1 generators. By performing this canonical transformation, the operator $\mathcal{L}(\lambda, \eta)$ is given in terms of spin-1 matrices $S^{\pm}$, S^z by the following expression

$$\mathcal{L}(\lambda, \eta) = \begin{pmatrix} \begin{matrix} \lambda I + f(\lambda,\eta) S^z - \\ \tilde{g}(\lambda,\eta)(S^z)^2 \end{matrix} & \frac{1}{\sqrt{2}}[\eta S^- S^z - 2\lambda h(\lambda,\eta) S^z S^-] & f(\lambda,\eta)(S^-)^2 \\ \begin{matrix} \frac{1}{\sqrt{2}}[\eta S^z S^+ - \\ 2\lambda h(\lambda,\eta) S^+ S^z] \end{matrix} & \begin{matrix} [\lambda - 3\eta h(\lambda,\eta)] I \\ +3\eta h(\lambda,\eta)(S^z)^2 \end{matrix} & \begin{matrix} -\frac{1}{\sqrt{2}}[\eta S^z S^- + \\ 2\lambda h(\lambda,\eta) S^- S^z] \end{matrix} \\ f(\lambda,\eta)(S^+)^2 & -\frac{1}{\sqrt{2}}[\eta S^+ S^z + 2\lambda h(\lambda,\eta) S^z S^+] & \begin{matrix} \lambda I - f(\lambda,\eta) S^z \\ -\tilde{g}(\lambda,\eta)(S^z)^2 \end{matrix} \end{pmatrix} \tag{5}$$

where I is the 3×3 identity matrix and functions $f(\lambda, \eta)$, $\tilde{g}(\lambda, \eta)$ and $h(\lambda, \eta)$ are given by

$$f(\lambda, \eta) = \eta \frac{(2\lambda - 3\eta/2)}{2(\lambda - 3\eta/2)}; \quad h(\lambda, \eta) = \frac{\eta}{2(\lambda - 3\eta/2)}; \quad \tilde{g}(\lambda, \eta) = 2\lambda + \frac{3\eta}{2} h(\lambda, \eta) \tag{6}$$

The Transfer matrix $T(\lambda, \eta)$ defined on the Hilbert space of L sites is the generator of commuting quantum integrals of motion. As usual, it is built up in terms of the vertex

operators by the expression

$$T(\lambda,\eta) = Tr_0[\mathcal{L}_{0L}(\lambda,\eta)\cdots\mathcal{L}_{01}(\lambda,\eta)] \tag{7}$$

where the index 0 stands for the 3×3 auxiliary space. The $\mathcal{L}(\lambda,\eta)$-matrix at $\lambda = 0$ is proportional to the operator of permutations and the logarithmic derivative of $T(\lambda,\eta)$ at this point defines the corresponding spin-1 Hamiltonian. The associated spin-1 chain is given by

$$H = \sum_{i=1}^{L} \Big[\frac{1}{3}\sigma_i^2 - \sigma_i - \frac{4}{3}[(\sigma_i^z)^2 - \sigma_i + \sigma_i^z\sigma_i^\perp + \sigma_i^\perp\sigma_i^z] + 2[(S_i^z)^2 + (S_{i+1}^z)^2] - \frac{7}{3}I \\ - \frac{1}{3}[\sigma_i^z(S_i^+S_{i+1}^- - S_i^-S_{i+1}^+) + (S_i^+S_{i+1}^- - S_i^-S_{i+1}^+)\sigma_i^z]\Big] \tag{8}$$

where $\sigma_i = S_i.S_{i+1} = \sigma_i^z + \sigma_i^\perp$ and $\sigma_i^z = S_i^z S_{i+1}^z$.

The first step toward the diagonalization of (8) is to notice that the vertex operator $\mathcal{L}(\lambda,\eta)$ resembles much that appearing in the construction of IK vertex model [12, 13, 16]. First of all, it is possible to verify that the operator $\mathcal{L}(\lambda,\eta)$ gives origin to 19 nonvanishing Boltzmann weights on a square lattice. This is schematized in Fig.1 by putting the values $\pm, 0$ on each bond of the lattice and by assuming a node current conservation. This picture suggests that a connection with the IK vertex model can be indeed tried. In fact, in Appendix B, we show that the Hamiltonian (8) can be obtained as an appropriate limit of that associated to the IK model. We also discuss how the algebraic Bethe ansatz method developed by Tarasov [15] can be directly applied to this problem. In the following we present the main steps of the analytical Bethe ansatz approach used in ref. [14]. We take as reference state the ferromagnetic vacuum defined by

$$|0> = \prod_i^L |0>_i; \quad |0>_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

The matrix $\mathcal{L}(\lambda,\eta)$ acting on the reference state has a triangular form with respect to

the auxiliary space, namely

$$\mathcal{L}(\lambda,\eta)|0>= \begin{pmatrix} \eta-\lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \frac{\lambda(\eta/2-\lambda)}{\lambda-3\eta/2} \end{pmatrix} \tag{10}$$

As a consequence, the corresponding eigenvalue $\Lambda(\lambda)$ of the Transfer matrix acting on this state is given by

$$\Lambda(\lambda) = (\eta-\lambda)^L + \lambda^L + \left[\frac{\lambda(\eta/2-\lambda)}{\lambda-3\eta/2}\right]^L \tag{11}$$

In order to construct other eigenvalues, the analytical approach seeks for a more general ansatz of form

$$\Lambda(\lambda,\{\lambda_j\}) = (\eta-\lambda)^L \prod_{j=1}^{M} A(\lambda_j-\lambda) + \lambda^L \prod_{j=1}^{M} B(\lambda_j-\lambda) + \left[\frac{\lambda(\eta/2-\lambda)}{\lambda-3\eta/2}\right]^L \prod_{j=1}^{M} C(\lambda_j-\lambda) \tag{12}$$

Following the arguments of ref. [14], crossing symmetry and unitarity condition of the vertex $\mathcal{L}(\lambda,\eta)$ and some analytical hypotheses concerning the behaviour of the Transfer matrix (7) fix functions $A(y)$, $B(y)$ and $C(y)$ to be

$$A(y) = \frac{\eta/2-y}{y+\eta/2}, \quad C(y-3\eta/2) = A^{-1}(y), \quad B(y) = A^{-1}(y+\eta/2)A(-y-\eta) \tag{13}$$

Collecting these results all together we finally find

$$\begin{aligned} \Lambda(\lambda,\{\lambda_j\}) = (i-\lambda)^L \prod_{j=1}^{M} -\frac{\lambda_j-\lambda-i/2}{\lambda_j-\lambda+i/2} + \lambda^L \prod_{j=1}^{M} \frac{\lambda_j-\lambda}{\lambda_j-\lambda+i}\frac{\lambda-\lambda_j-3i/2}{\lambda-\lambda_j-i/2} \\ + \left[\frac{\lambda(i/2-\lambda)}{\lambda-3i/2}\right]^L \prod_{j=1}^{M} -\frac{\lambda-\lambda_j-2i}{\lambda-\lambda_j-i} \end{aligned} \tag{14}$$

where due to the scale invariance $\mathcal{L}(\eta\lambda,\eta) = \eta\mathcal{L}(\lambda,1)$ we have chosen the parameter $\eta = i$.

The set of numbers $\{\lambda_j\}$ are then fixed by imposing that function $\Lambda(\lambda,\{\lambda_j\})$ has no pole at finite value of λ. This means that the residues of $\Lambda(\lambda,\{\lambda_j\})$ at the poles $\lambda = \lambda_j+i/2$ and $\lambda = \lambda_j+i$ must vanish. An important check of the ansatz (14) is that these pole conditions should give the same restriction for the set $\{\lambda_j\}$. In fact, this is guaranteed

by the crossing symmetry of the operator $\mathcal{L}(\lambda, \eta)$ and we find the following Bethe ansatz equation

$$\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L = -(-1)^r \prod_{k=1}^{M} \left(\frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}\right) \left(\frac{\lambda_j - \lambda_k + i/2}{\lambda_j - \lambda_k - i/2}\right) \tag{15}$$

where $r = L - M$ [2] . The spin chain (8) commutes with the $U(1)$ charge and the index r labels the disjoint sectors of the theory with magnetization $r = \sum_i^L S_i^z$. The eigenenergies $E^r(L)$ of such Hamiltonian in a given sector r is obtained by taking the logarithmic derivative of $\Lambda(\lambda, \{\lambda_j\})$ at $\lambda = 0$, namely

$$E^r(L) = -\sum_{j=1}^{L-r} \frac{1}{\lambda_j^2 + 1/4} + L \tag{16}$$

We believe that an important feature of equation (15) is the presence of the phase factor $(-1)^r$ distinguishing the behaviour of solutions $\{\lambda_j\}$ in the odd and even sectors of the theory. Hence, the bare phase-shift at equal rapidities $\lambda_j = \lambda_k$ can assume both positive and negative values, depending on the sector r. These signs may be connected [20] to the different possibilities of exchanging (statistical behaviour) the excitations (periodic/antiperiodic boundary conditions of the wave function) in the model. The physical consequence of this fact is the evidence of an explicit separation between the fermionic and bosonic degrees of freedom of the system. These observations strongly indicate the presence of an extra symmetry in the spin-1 chain (8). In fact, the $Osp(1|2)$

[2] At this point we recall that in ref. [3], by using a different approach of ours, the author presents a Bethe ansatz equation without the phase factor $-(-1)^r$. We stress that his discussion is very brief and at least for us it is not clear which boundary condition has been taken into account. However, in Appendix C, we have considered a quite general twisted boundary condition compatible with the $Osp(1|2)$ invariance and it is still noted the presence of such phase factor. Moreover, as it has been discussed in Appendix C, the absence of the factor $-(-1)^r$ indicates an inconsistency with the expected degeneracy of a $Osp(1|2)$ invariant system. Hence, this forces us to conclude that the Bethe ansatz equation without the factor $-(-1)^r$ either represents only the odd part of the spectrum and therefore is incomplete or is related to some peculiar boundary condition incompatible with the $Osp(1|2)$ symmetry . In any case, the underlying quantum field theory should then be different from that found in section 4, since that such factor is crucial in the computation of the corresponding critical exponents.

chain (besides the $U(1)$ charge) commutes with the even generators generated by $\tilde{S}^{\pm}$ (see Appendix A). The spin-1 version of this invariance is the following commutation relation

$$[H, \sum_{i=1}^{L} (S_i^{\pm})^2] = 0 \tag{17}$$

The conserved charge (17) implies that sectors differing of step ± 2 can share common eigenvalues. Such symmetry will be extremely important in the characterization of the finite size properties to be discussed in the next sections.

3 The completeness of the Bethe ansatz for L=4 sites

This section is concerned with the completeness of the Bethe ansatz equation (15) for $L = 4$ and consequently with the numerical study of the behaviour of roots $\{\lambda_j\}$ for a $finite$ number of sites. This analysis may lead us to discover interesting structures of roots $\{\lambda_j\}$ and also to verify whether or not these solutions are complete. Such study is motivated by the appearance of a peculiar bare phase shift (right hand side of (15)) in the Bethe ansatz equations. In fact, at first glance one already notices that, in the even sector, the phase $-(-1)^r$ [3] prohibits symmetric solutions $\{\lambda_j\}$ possessing one of the roots at the origin.

In Table 1, for $L = 4$, we present the possible configurations of zeros $\{\lambda_j\}$ and their corresponding eigenvalues of energy and momenta. This has been done by numerically solving the Bethe ansatz equations (15) and comparing them to exact diagonalization of the $Osp(1|2)$ chain. In our notation, $\{n\}$ refers to the possible standard n-string [4] structures appearing as a solution of (15). The subscript k in n_k is the integer or half-integer number which better represents the logarithmic branches of equation (15) for the real part of solution $\{\lambda_j\}$ (see e.g. equation (18)). The first consequence of this study is

[3] We notice that similar phase sign has previously appeared in the $SL(1|1)$ model (spin-1/2 XX chain), see e.g. ref. [7].

[4] We recall that a n-string is characterized by the root $\lambda_j^{n,\alpha} = \chi_j^n + \frac{i}{2}(n + 1 - 2\alpha)$ $\alpha = 1, 2, \cdots, n$; where χ_j is a real number and n is the length of the string.

to show that the Bethe ansatz solutions produce the complete spectrum of the $Osp(1|2)$ spin chain, at least for $L = 4$. Here, we have evidently taken advantage of the hidden symmetry discussed in section 2 (equation 17), by separating the even and odd sectors of the theory. Previous experience with other spin chains [21] would suggest that these results are a strong indication of the completeness of the roots $\{\lambda_j\}$ even in the thermodynamic limit.

In Table 2 we explicitly present some of the roots $\{\lambda_j\}$ in order to exemplify our notation of Table 1. As it has already been noticed, we have observed that the even sector does not admit a root (symmetric) exact on the origin. Instead, they prefer to form anomalous string configurations which have been characterized by the symbols A and B. The first structure A involves four zeros and one may think that in the thermodynamic limit they would lead to a 3-string plus one 1-string at the origin. By solving the Bethe ansatz equations for several values of $L > 4$, we have verified that this is possible, but corresponds to a higher excited state on the spectrum. On the contrary, a lower excitation is produced when the smaller imaginary part of structure A grows and its bigger imaginary part decreases. This effect is shown in Table 3 for lattices up to 12 sites [5]. Definitely, this anomaly can not be understood in terms of the usual string formulation. Analogously, we have verified that the structure B does not go to $\pm i1.5$ and thus, together with $2_0^\dagger$, forming a 4-string. Again, for the lowest excitation the imaginary part of B decreases.

In general, we have verified that for higher L the root system is in fact plagued with such anomalous string structures. Moreover, in the course of our Bethe ansatz computations we have noticed an interesting resemblance to certain properties of the $O(3)$ invariant spin chain. Some years ago, the author [22] has shown that fractional strings do appear in this model. This is an indication that the usual string hypothesis has to be modified for the $Osp(1|2)$ chain. So far a precise reformulation of this hypothesis has eluded us. However, we hope to return to this matter, since we believe that this new structure will play an

[5] For a quantitative analysis of the finite size properties one has to go beyond $L > 12$ in order to take into account possible logarithmic corrections .

important role in the thermodynamic properties.

4 The thermodynamic limit and the finite-size behaviour

We start this section by investigating the thermodynamic limit of the Bethe ansatz equation (15). We shall concentrate our analysis in the ground state of a given sector r. In this case the solutions $\{\lambda_j\}$ are real [6] and by taking the logarithmic of equation (15) we find

$$L\psi_{1/2}(\lambda_j) = 2\pi Q_j + \sum_{k=1,k\neq j}^{L-r} [\psi_1(\lambda_j-\lambda_k) - \psi_{1/2}(\lambda_j-\lambda_k)] \tag{18}$$

where $\psi_a(x) = 2\arctan(x/a)$ and Q_j are integer or semi-integer numbers defining the different branches of the logarithm. For those states we find

$$Q_j = -\frac{[L-r-1]}{2} + j - 1, \quad j = 1, 2, \cdots, L-r \tag{19}$$

For large L, the roots tend toward a continuous distribution with density $\rho_L^r(\lambda)$ given by

$$\rho_L^r(\lambda) = \frac{d}{d\lambda} Z_L^r(\lambda) \tag{20}$$

where the counting function $Z_L^r(\lambda)$ [23] is defined by

$$Z_L^r(\lambda_j) = \frac{Q_j^r}{L} = \frac{1}{2\pi}\left\{\psi_{1/2}(\lambda_j) - \frac{1}{L}\sum_{k=1,k\neq j}^{L-r}[\psi_1(\lambda_j-\lambda_k - \psi_{1/2}(\lambda_j-\lambda_k)]\right\} \tag{21}$$

Strictly in the thermodynamic limit ($L \to \infty$), the system (20,21) goes into an integral equation for the density $\rho_\infty(\lambda)$ given by

$$2\pi\rho_\infty(\lambda) + \int_{-\infty}^{\infty}[\psi_1'(\lambda-\mu) - \psi_{1/2}'(\lambda-\mu)]\rho_\infty(\mu)d\mu = \psi_{1/2}'(\lambda) \tag{22}$$

[6] This structure has been determined by numerically solving the Bethe ansatz equations (15) for many values of the lattice size L.

where the prime symbol stands for the derivative. This equation is then solved by using the Fourier transform method and we find

$$\rho_\infty(\lambda) = \frac{2}{\sqrt{3}} \frac{\cosh(2\pi\lambda/3)}{\cosh(4\pi\lambda/3) + 1/2} \tag{23}$$

and from equation (16) the ground state energy per site e_∞ is calculated to be

$$e_\infty = -\int_{-\infty}^{\infty} \frac{\rho_\infty(\lambda)}{\lambda^2 + 1/4} d\lambda + 1 = -\frac{4\pi\sqrt{3}}{9} + 1 \cong -1.4184... \tag{24}$$

Now we turn to the finite size corrections of the lowest energies $E^r(L)$ of a given sector r. Our computation will be based on a method introduced by De Vega and Woynarovich [23] and further developed in order to be applied for spin chains [24, 25] and to the Hubbard model [26, 27]. Using this approach we are able to write analytical expressions for the difference of the energies and density of roots from their corresponding bulk values. Following ref. [23] we have

$$\frac{E^r(L)}{L} - e_\infty = -2\pi \int_{-\infty}^{\infty} \rho_\infty(\mu) S_L^r(\mu) d\mu \tag{25}$$

and

$$\rho_L^r(\lambda) - \rho_\infty(\lambda) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} p(\lambda - \mu) S_L^r(\mu) d\mu \tag{26}$$

where function $S_L^r(\mu)$ and the Fourier transform of function $p(x)$ are defined by

$$S_L^r(\mu) = \frac{1}{L} \sum_{j=1}^{L-r} [\delta(\lambda_j - \mu) - \rho_L^r(\mu)] \tag{27}$$

$$[1 - p(\omega)]^{-1} = G_+(\omega) G_-(\omega) = \frac{e^{-|\omega|/2} \Gamma(1/2 - i\omega/4\pi) \Gamma(1/2 + i\omega/4\pi)}{\Gamma(1/2 - i3\omega/4\pi) \Gamma(1/2 + i3\omega/4\pi)} \tag{28}$$

where the Fourier transform of $p(x)$ is defined as $p(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\omega} p(x) dx$. According to this technique, the first order corrections can be calculated with the help of the Euler-Maclarium formula and equations (25,26) can be rewritten up to order $O(1/L^2)$ as

$$\frac{E^r(L)}{L} - e_\infty = 4\pi \left\{ \int_{\Lambda}^{\infty} \rho_\infty(\lambda) \rho_L^r(\lambda) d\lambda - \frac{\rho_\infty(\Lambda)}{2L} - \frac{\rho'_\infty(\Lambda)}{12 L^2 \rho_L^r(\Lambda)} \right\} \tag{29}$$

where the density $\rho_L^r(\lambda+\Lambda)$ satisfies the following Wiener-Hopf integral equation

$$X^r(t) = \rho_\infty(\lambda) + \frac{1}{2\pi}\left\{\int_0^\infty X^r(t)p(t-\mu)d\mu - \frac{p(t)}{2L} - \frac{p'(t)}{12L^2}\right\} \tag{30}$$

where $t = \lambda - \Lambda$,$X^r(\lambda) = \rho_L^r(\lambda+\Lambda)$ and Λ is the largest magnitude root determined by the boundary condition

$$\int_\Lambda^\infty \rho_L^r(\lambda)d\lambda = \frac{1}{2L} + \frac{r}{2L} \tag{31}$$

This integral equation is solved by introducing the Fourier transform

$$X_\pm^r(\omega) = \int_{-\infty}^\infty e^{i\omega t}X_\pm^r(t); \quad X_\pm^r(t)\begin{cases} X^r(t) & t >_< 0 \\ 0 & t <_> 0 \end{cases} \tag{32}$$

and after some algebra (see e.g. [25]) we find

$$X_+^r(\omega) = C^r(\omega) + G_+(\omega)[Q_+(\omega) + P(\omega)] \tag{33}$$

where

$$C^r(\omega) = \frac{1}{2L} - \frac{i\omega}{12L^2\rho_L^r(\omega)}, \quad Q_+(\omega) = \frac{2}{\sqrt{3}}\frac{G_-(-i2\pi/3)}{2\pi/3 - i\omega}e^{-2\pi\Lambda/3} \tag{34}$$

$$P(\omega) = -\frac{1}{2L} + \frac{ig}{12L^2\rho_L^r(\Lambda)} - \frac{i\omega}{12L^2\rho_L^r(\Lambda)}, \quad g = -\frac{1}{9} \tag{35}$$

Finally, using all these results in equation (29) and approximating $\rho_\infty(\Lambda) \simeq \frac{2}{\sqrt{3}}e^{-2\pi\Lambda/3}$ we obtain the first correction for the lowest energy sector as

$$\frac{E^r(L)}{L} - e_\infty = \frac{\pi^2\xi}{L^2}(-\frac{1}{6} + \frac{r^2}{2G_+(0)^2}) \tag{36}$$

where the sound velocity is calculated to be $\xi = 2\pi/3$ [17] and $G_+(0)^2 = 1$.

Considering the predictions of conformal invariance for a finite system of size L, this last result leads to a central charge $c = 1$. The conformal dimensions X_r associated to the lowest state on the sector r are

$$X_r = \frac{r^2}{4} \tag{37}$$

This operator content has to be understood in the context of a Gaussian model with a coupling constant proportional to 1/4. In general, besides the " spin-wave " state

r, we shall expect that the complete operator content has also a "vortex " excitation parametrized by the index m. The conformal dimensions are then given by

$$X_{r,m} = \frac{r^2}{4} + m^2 \tag{38}$$

Such conformal dimensions are in accordance with the hidden symmetry discussed in section 2. For instance, even excitations on certain sector $r = 2n$ have to be included on the zero sector with dimensions n^2. Indeed from (38) we see that $X_{2n,0} = X_{0,n}$.

5 Concluding remarks

We have shown that the quantum $Osp(1|2)$ spin chain is solvable by the Bethe ansatz approach. In particular, we find that the Bethe ansatz equations present a new property of explicitly distinguishing the even and odd sectors of the theory. This feature, as discussed in Appendix A, is a direct consequence of an extra symmetry. Remarkably enough, such symmetry resembles much that of a multiplicative fermionic parity appearing in supersymmetric field theories. Analogously, this index can project out the fermionic and bosonic parts of the superfield. In our model, this invariance has an important influence on the finite size effects, even inducing the appearance of new kind of string structures.

After an appropriate reformulation, the $Osp(1|2)$ spin chain can be seen as certain limit of the Izergin-Korepin [12, 13, 16] vertex model. As a consequence, we are able to show that the Izergin-Korepin system admits an extra isotropic solution besides the known $SU(3)$ invariant point [14]. Due to this novel property, we believe that our formulation of the Izergin-Korepin model presented in Appendix B is the most natural one for studying the bosonic/fermionic splitting in a reduced $c = 1$ conformal field theory. The simplest reduction should be the tricritical Ising model which present a Neveu-Schwartz and Ramond sectors of excitation [28]. Hopefully, the study of our spin chain with both periodic and antiperiodic boundary conditions will be able to select the even and odd sectors of the theory. Work on this direction is in progress.

The results of this paper also suggest to look for a more general spin Hamiltonian possessing the $Osp(1|2)$ invariance. One possibility is as follows

$$H = \sum_{i=1}^{L} \{J_1 C_{i,i+1} + J_2 C^2_{i,i+1}\} \tag{39}$$

where J_1, J_2 are free parameters and $C_{i,i+1}$ is the $Osp(1|2)$ Casimir operator (see Appendix A). This theory has at least three integrable points. The point $J_2/J_1 = 5/9$ is the critical $Osp(1|2)$ chain. At $J_2/J_1 = 1/3$ the Hamiltonian is proportional to the graded permutation operator, thus possessing an $Sl(1|2)$ symmetry [5, 30]. The ground state is ferromagnetic and the excitations are gapless. The third point is $J_1 = 0$ and the model is just the $Osp(1|2)$ Temperely-Lieb operator [18]. In this case the Temperely-Lieb parameter is negative and the model is still massless. Indeed one can show its correspondence with an appropriate point of the deformed biquadratic spin-1 chain [18]. Therefore, in contrast to the bilinear biquadratic $SU(2)$ invariant spin-1 chain (see e.g. ref. [31]), all integrable points seem to present only massless degrees of freedom. It should be an interesting problem to discuss the phase diagram of the Hamiltonian (39) and in particular to verify whether or not there exists massive regimes.

Finally, we recall that it is possible to construct more general R-matrices presenting a $Osp(n|2m)$ invariance [4, 19, 18, 34, 35, 36] . Then, one would like to ask if the rich feature found in the spectrum of the $Osp(1|2)$ chain can be even more general for these other systems. Our recent results on the Bethe ansatz for the $Osp(1|2n)$ chain [29] and that of ref [36] for the $Osp(2|2)$ model strongly indicate that interesting properties are still to be discovered.

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Appendix A

The purpose of this appendix is to write the Hamiltonian (3) in terms of the Casimir operator of the $Osp(1|2)$ algebra. Such operator has the following expression [37]

$$C_{i,i+1} = 4\tilde{S}^z_i \overset{s}{\otimes} \tilde{S}^z_{i+1} + 2[\tilde{S}^+_i \overset{s}{\otimes} \tilde{S}^-_{i+1} + \tilde{S}^-_i \overset{s}{\otimes} \tilde{S}^+_{i+1}] + 4[V^+_i \overset{s}{\otimes} V^-_{i+1} - V^-_i \overset{s}{\otimes} V^+_{i+1}] \tag{A.1}$$

where the even (bosonic) generators $\tilde{S}^{\pm}, \tilde{S}^z$ are

$$\tilde{S}^z = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad \tilde{S}^+ = (\tilde{S}^-)^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

and the odd (fermionic) operators $V^{\pm}$ are

$$V^+ = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \quad V^- = \begin{pmatrix} 0 & 0 & 0 \\ -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \tag{A.3}$$

In equation (A.1) the symbol $\overset{s}{\otimes}$ stands for the supertensor product between two matrices. More precisely we have

$$(A \overset{s}{\otimes} B)^{ij}_{ab} = (-1)^{p(i)p(j)+p(a)p(b)+p(i)p(B)} A_{ai} B_{bj} \tag{A.4}$$

where $p(f)$ is the Grassmann parity of the object f (vector index or matrix). In our case we have the graduation FBF for the space index $i = 1, 2, 3$, $p(\tilde{S}^{\pm}) = p(\tilde{S}^z) = 0$ and $p(V^{\pm}) = 1$. Using equations (A.1-3) and the latest definitions, the Casimir operator is

rewritten as a 9 matrix of form

$$C_{i,i+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.5}$$

In order to make the connection between this operator and the Hamiltonian (3) one has to redefine the grading to BFF. After this canonical transformation we find the fundamental relation

$$C_{i,i+1} = E^g_{i,i+1} - P^g_{i,+1} \tag{A.6}$$

Finally, by using equation (A.6) we can write the spin chain as

$$H = \sum_{i=1}^{L} \{C_{i,i+1} + \frac{5}{9}(C_{i,i+1})^2 - \frac{5}{9}\} \tag{A.7}$$

where we have used the important braid-monoid properties $E_g^2 = -E_g$ and $P_g E_g = E_g P_g$ proved in ref. [18]. From all these results it is possible to find the following commutation relations

$$[H, \sum_{i=1}^{L} \tilde{S}_i^z] = 0 \tag{A.8}$$

$$[H, \sum_{i=1}^{L} \tilde{S}_i^{\pm}] = 0 \tag{A.9}$$

Appendix B

Here we first discuss the relation between the $Osp(1|2)$ and the Izergin-Korepin vertex

model [12, 13, 16]. In order to do that we have to reformulate conveniently the Boltzmann weights of the IK system. The R-matrix in which we are interested can be written as

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c(\lambda) & 0 & e^{+}(\lambda) & 0 & f^{+}(\lambda) & 0 & 0 \\ 0 & 1 & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-}(\lambda) & 0 & g(\lambda) & 0 & e^{+}(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & 1 & 0 \\ 0 & 0 & f^{-}(\lambda) & 0 & e^{-}(\lambda) & 0 & c(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) \end{pmatrix} \tag{B.1}$$

where

$$a(\lambda) = \frac{\sin[\gamma(i\lambda+1)]}{\sin[\gamma]}, \quad b(\lambda) = -\frac{\sin[i\gamma\lambda]}{\sin[\gamma]}, \quad c(\lambda) = \frac{\sin[i\gamma\lambda]\sin[\gamma(i\lambda+1/2)]}{\sin[\gamma]\sin[\gamma(i\lambda+3/2)]} \tag{B.2}$$

and

$$e^{\pm} = \mp i e^{\mp i\gamma}\frac{\sin[i\gamma\lambda]}{\sin[\gamma(i\lambda+3/2)]}, \; f^{\pm} = a(\lambda) - e^{\mp i2\gamma}c(\lambda), \; g(\lambda) = -\frac{\sin[i\gamma\lambda]}{\sin[\gamma]} + \frac{\sin[3\gamma/2]}{\sin[\gamma(i\lambda+3/2)]} \tag{B.3}$$

It is easy to check that this solution satisfies the properties appearing in the IK vertex model [12, 14, 15]. For instance, to recover the original notation of ref [15] one has to roughly replace $\lambda/2 \to \lambda$ and $\eta \to \pi/2 + i\gamma/2$. We also have to use some symmetries of this solution, such as $b \to -b$ and $e^{\pm} \to \pm i e^{\pm}$. The Hamiltonian associated with the vertex model (B.1) is calculated to be

$$\begin{aligned} H = \tilde{J}\sum_{i=1}^{L} \Big[& \sin(\gamma/2)\sigma_i^2 - \sin(3\gamma/2)\sigma_i - 2\sin(\gamma)\cos(3\gamma/2)[(\sigma_i^z)^2 - \sigma_i^z] \\ & -2\sin(\gamma)\cos(\gamma/2)(1-\sin(\gamma/2))[\sigma_i^z\sigma_i^{\perp} + \sigma_i^{\perp}\sigma_i^z] + 2\sin(3\gamma/2)\cos^2(\gamma/2)[(S_i^z)^2 + (S_{i+1}^z)^2] \\ & -[\sin(\gamma/2) + 2\cos^2(\gamma/2)\sin(3\gamma/2)]I + \frac{i}{2}\sin(\gamma/2)\sin(2\gamma)[(S_i^z)^2 S_{i+1}^z - S_i^z(S_{i+1}^z)^2] \\ & -i\frac{\sin(2\gamma)}{4}[(S_{i+1}^z - S_i)\sigma_i^{\perp} + \sigma_i^{\perp}(S_{i+1}^z - S_i^z)\Big] \end{aligned} \tag{B.4}$$

where $\tilde{J} = J\frac{\gamma}{\sin(\gamma)\sin(3\gamma/2)}$. Such Hamiltonian can be diagonalized by using the analytical approach of section 2 or by directly applying the algebraic calculation of ref [15]. As a final result we find that the spectrum is parametrized by the following Bethe ansatz equations

$$\left(\frac{\sinh[\gamma(\lambda_j - i/2)]}{\sinh[\gamma(\lambda_j + i/2)]}\right)^L = -(-1)^r \prod_{k=1}^{M} \frac{\sinh[\gamma(\lambda_j - \lambda_k - i)]}{\sinh[\gamma(\lambda_j - \lambda_k + i)]} \frac{\sinh[\gamma(\lambda_j - \lambda_k + i/2)]}{\sinh[\gamma(\lambda_j - \lambda_k - i/2)]} \tag{B.5}$$

where the eigenenergies are given by

$$E^r(L) = \sum_{i=1}^{L-r} \frac{2\gamma\sin(\gamma)}{\cos(\gamma) - \cosh(2\gamma\lambda_j)} + \frac{\gamma\cos(\gamma)}{\sin(\gamma)} \tag{B.6}$$

Hence, taking the limit $\gamma \to 0$ and collecting factors up to first order in γ we then recover the isotropic spin-1 Hamiltonian (8). It is crucial to notice that the last term of equation (B.4) and (8) are equivalent by a canonical transformation of type $e^{\pm} \to \pm ie^{\pm}$. Moreover, from equations (B.5, B.6) the limit $\gamma \to 0$ directly recovers the Bethe equations of $Osp(1|2)$ chain.

At this point we recall that the quantum spin chain associated with the IK vertex model has been recently discussed in the literature (see e.g. [38, 39] and references therein). However, in all cases the problem has been formulated in such way that one obtains the $SU(3)$ invariant chain as the isotropic point. Remarkably enough, this other isotropic branch limit ,the $Osp(1|2)$ chain discussed in this paper, has not been noticed before[7]. Thus , in our formulation the IK model is seen as a certain deformation of the $Osp(1|2)$ chain. We believe that this is the most appropriate formulation of the problem, in order to get reduced conformal theories presenting both fermionic and bosonic degrees of freedom.

Now we are going to summarize the solution of the $Osp(1|2)$ chain by the algebraic Bethe ansatz approach developed by Tarasov [15]. In this case one works directly with

[7] For instance, considering notation of ref [38] one must do the shift $\gamma \to \pi - \gamma$ before making the limit $\gamma \to 0$.

the operator content of the monodromy matrix $\tau(\lambda)$

$$\tau(\lambda) = \mathcal{L}_{0L}(\lambda) \cdots \mathcal{L}_{01}(\lambda) = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & B_2(\lambda) \\ C_1(\lambda) & A_2(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_3(\lambda) & A_3(\lambda) \end{pmatrix} \tag{B.7}$$

and with the integrability condition

$$\tilde{\mathcal{L}}(\lambda - \mu)\tau(\lambda) \otimes \tau(\mu) = \tau(\mu) \otimes \tau(\lambda)\tilde{\mathcal{L}}(\lambda - \mu) \tag{B.8}$$

where $\tilde{\mathcal{L}}_{ab}^{cd}(\lambda) = \mathcal{L}_{ba}^{cd}(\lambda)$.

On the reference state $|0>$ we have

$$T(\lambda)|0>= \sum_{i=1}^{3} A_i|0>= (i-\lambda)^L + \lambda^L + \left[\frac{\lambda(i/2-\lambda)}{(\lambda - 3i/2)}\right]^L \tag{B.9}$$

The one particle excitation $|\phi(\lambda_1)>$ over the pseudovacuum is given by

$$|\phi(\lambda_1)>= B_1(\lambda_1)|0> \tag{B.10}$$

In order to calculate the Transfer matrix acting on the one particle state one has to take advantage of the following commutation relations coming from equation (B.8)

$$A_1(\lambda)B_1(\lambda_1)|0>= f_1(\lambda_1 - \lambda)B_1(\lambda_1)A_1(\lambda)|0> -f_2(\lambda_1 - \lambda)B_1(\lambda)A_1(\lambda_1)|0> \tag{B.11}$$

$$\begin{aligned} A_2(\lambda)B_1(\lambda_1)|0>= \frac{f_1(\lambda - \lambda_1)}{f_3(\lambda - \lambda_1)}B_1(\lambda_1)A_2(\lambda)|0> &+f_2(\lambda_1 - \lambda)B_1(\lambda)A_2(\lambda_1)|0> \\ &+f_4(\lambda - \lambda_1)B_3(\lambda)A_1(\lambda_1)|0> \end{aligned} \tag{B.12}$$

$$A_3(\lambda)B_2(\lambda_1)|0>= f_5(\lambda - \lambda_1)B_1(\lambda_1)A_3(\lambda)|0> -f_4(\lambda - \lambda_1)B_3(\lambda)A_2(\lambda_1)|0> \tag{B.13}$$

where

$$f_1(x) = \frac{i-x}{x};\ \ f_2(x) = \frac{i}{x};\ \ f_3(x) = \frac{i/2 - x}{i/2 + x};\ \ f_4(x) = \frac{i}{i/2 - x};\ f_5(x) = \frac{x - 3i/2}{i/2 - x} \tag{B.14}$$

By using these relations we find that

$$T(\lambda)|\phi(\lambda_1)>= \sum_{i=1}^{3} A_i(\lambda)B_1(\lambda_1)|0>= \Lambda(\lambda, \lambda_1)|\phi(\lambda_1)> \tag{B.15}$$

where

$$\Lambda(\lambda,\lambda_1) = (i-\lambda)^L \frac{i-\lambda_1+\lambda}{\lambda_1-\lambda}\lambda^L \frac{i-\lambda+\lambda_1}{\lambda-\lambda_1}\frac{i/2-\lambda_1+\lambda}{i/2+\lambda_1-\lambda} + \left[\frac{\lambda(i/2-\lambda)}{\lambda-3i/2}\right]^L \frac{3i/2-\lambda+\lambda_1}{\lambda-\lambda_1-i/2} \tag{B.16}$$

provided that

$$\left(\frac{i-\lambda_1}{\lambda_1}\right)^L = 1 \tag{B.17}$$

The two particle state is given by the ansatz [15]

$$|\phi(\lambda_1,\lambda_2) >= [B_1(\lambda_1)B_2(\lambda_2) + \eta(\lambda_1,\lambda_2)B_2(\lambda_1)A_1(\lambda_2)]|0> \tag{B.18}$$

where function $\eta(\lambda_1,\lambda_2)$ is determined by imposing that under permutation $\lambda_1 \leftrightarrow \lambda_2$ the symmetric state $|\phi(\lambda_2,\lambda_1) >$ is at most proportional to that of equation (B.18) . This is solved by using the commutation relation

$$B_1(\lambda_2)B_1(\lambda_1) = f_3(\lambda_1-\lambda_2)[B_1(\lambda_1)B_1(\lambda_2)-f_4(\lambda_1-\lambda_2)B_2(\lambda_1)A_1(\lambda_2)]+f_4(\lambda_2-\lambda_1)B_2(\lambda_2)A_1(\lambda_1) \tag{B.19}$$

and we find that $\eta(\lambda_1,\lambda_2) = f_4(\lambda_1-\lambda_2)$. Moreover, the Transfer matrix on this symmetrized state has the following eigenvalue

$$\Lambda(\lambda,\lambda_1,\lambda_2) = (i-\lambda)^L \prod_{j=1}^{2} \frac{i-\lambda_j+\lambda}{\lambda_j-\lambda} + \lambda^L \prod_{j=1}^{2} \frac{i-\lambda+\lambda_j}{\lambda-\lambda_j}\frac{i/2-\lambda_j+\lambda}{i/2+\lambda_j-\lambda} + \left[\frac{\lambda(i/2-\lambda)}{\lambda-3i/2}\right]^L \prod_{j=1}^{2} \frac{3i/2-\lambda+\lambda_j}{\lambda-\lambda_j-i/2} \tag{B.20}$$

provided that

$$\left(\frac{i-\lambda_j}{\lambda_j}\right)^L = -\prod_{k=1}^{2} \frac{\lambda_j-\lambda_k-i}{\lambda_j-\lambda_k+i}\frac{\lambda_j-\lambda_k+i/2}{\lambda_j-\lambda_k-i/2} \tag{B.21}$$

Following the arguments of ref. [15] these last results can be generalized for any n-particle excitation. We then recover equation (15) by making the shift $\lambda_j \to \lambda_j + i/2$.

Appendix C

This appendix is concerned with the study of the $Osp(1|2)$ chain with twisted boundary condition. It is possible to check that the $Osp(1|2)$ algebraic structure is invariant under the following twisted transformation :

$$V^{\pm} \to e^{\pm i\phi/2} V^{\pm}, \quad \tilde{S}^{\pm} \to e^{\pm i\phi} \tilde{S}^{\pm}, \quad \tilde{S}^{z} \to \tilde{S}^{z} \tag{C.1}$$

This means that one can define a more general boundary condition (preserving the $Osp(1|2)$ algebra) by

$$V^{\pm}_{L+1} \to e^{\pm i\phi/2} V^{\pm}_{1}, \quad \tilde{S}^{\pm}_{L+1} \to e^{\pm i\phi} \tilde{S}^{\pm}_{1}, \quad \tilde{S}^{z}_{L+1} \to \tilde{S}^{z}_{1} \tag{C.2}$$

At this point it is interesting to remark that, besides the usual periodic case ($\phi = 0$), the condition $(C.2)$ admits the interesting mixed case of a periodic (antiperiodic) boundary condition in the bosonic (fermionic) degrees of freedom by taking the angle $\phi = 2\pi$. By imposing the boundary $(C.2)$ and from Appendix A we find that the boundary term $H_b(\phi)$ of the corresponding $Osp(1|2)$ Hamiltonian is given by

$$H_b(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i\phi/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & \frac{2}{3}e^{-i\phi/2} & 0 & \frac{1}{3}e^{-i\phi} & 0 & 0 \\ 0 & -e^{i\phi/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3}e^{i\phi/2} & 0 & -5/3 & 0 & \frac{2}{3}e^{-i\phi/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-i\phi/2} & 0 \\ 0 & 0 & \frac{1}{3}e^{i\phi} & 0 & -\frac{2}{3}e^{i\phi/2} & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{i\phi/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.3}$$

Moreover, by performing the Bethe ansatz analysis of Appendix B we find that the associated Bethe ansatz equation of the $Osp(1|2)$ chain with twisted boundary condition $(C.2)$ is

$$\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L = -(-1)^r e^{i\phi/2} \prod_{k=1}^{M} \left(\frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i}\right) \left(\frac{\lambda_j - \lambda_k + i/2}{\lambda_j - \lambda_k - i/2}\right) \tag{C.4}$$

and the energy equation (16) remains unchanged.

The importance of this analisis is that it allows us to study the behaviour of the spectrum of a theory possessing Bethe ansatz equation (15) without the presence of the factor $-(-1)^r$. This is as follows. For odd sectors $-(-1)^r = +1$, therefore the behaviour of the spectrum is precisely the same found for the $Osp(1|2)$ with periodic boundary condition. For even sectors, however, this can be studied by varying adiabaticaly the angle ϕ up to $\phi = 2\pi$. For instance, in Table 4, we present the ground state structure of roots λ_j for $L = 2$ and $r = 0$. We observe that when $\phi \to 2\pi$ one of the roots diverges and the other goes to zero. Moreover, when $\phi \to 2\pi$, we have numerically verified for several values of L that the usual set of L roots $\{\lambda_1(\phi), \cdots, \lambda_L(\phi)\}$ of the ground state of sector $r = 0$ goes to $\{\infty, 0, \lambda_3, \cdots, \lambda_L\}$ where the set $\{0, \lambda_3, \cdots, \lambda_L\}$ is precisely the roots characterising the ground state of sector $r = 1$. This means that in absence of the phase factor $-(-1)^r$ ($\phi = 2\pi$ and $r = 0$) the ground state of the sectors $r = 0$ and $r = 1$ are degenerated, since the root $\lambda = \infty$ gives a null contribution to the energy. Of course the same reasoning can be repeated for any pair of sectors $r = 2n$ and $r = 2n + 1$, $n = 0, 1, \cdots$. As a consequence, we conclude that (in absence of the factor $-(-1)^r$) the degeneracy is of step ± 1 which is in contradiction to that found (step ± 2) for a $Osp(1|2)$ invariant system (see Appendix A, Eq.(A.9)). Nevertheless, due to this degeneracy, some conformal dimensions found in section 4 are now prohibited changing remarkably the underlying quantum field theory. Finally, we recall that the presence of the factor $-(-1)^r$ changes the topology of the Bethe ansatz equation for sector $r = 1$ and no such contradiction is found. Hopefully, this analysis will lend further support to the physical meaning and to the importance of the phase factor $-(-1)^r$ in the Bethe ansatz equation.

References

[1] B. Sutherland, *Phys.Rev.B 12 (1975) 3795*

[2] P.P. Kulish and E.K. Sklyanin, *J.Sov.Math.19 (1982) 1596*; Lectures Notes in Physics 151 p.61, edited by J. Hietarinta and C. Montonen (Springer-Verlag)

[3] P.P. Kulish, *J.Sov.Math.35 (1986) 2648*

[4] V.V. Bazhanov and A.G. Shadrikov, *Theor.Math.Phys.73 (1987) 1302*

[5] C.L. Schultz, *Physica A 122 (1983) 71*

[6] J.H.H. Perk and C.L. Schultz, *Non-Linear Integrable Systems-Classical and Quantum Theory, ed M. Jimbo and T. Miwa (World Scientific, Singapore) p135*

[7] H.J. de Vega and E. Lopes, *Phys.Rev.Lett.67 (1991) 489*

[8] F.H.L. Essler and V.E. Korepin, *Phys.Rev.B 46 (1992) 9147*

[9] F.H.Essler, V.E. Korepin and K. Schoutens, *Phys.Rev.Lett.68 (1992) 2960, Phys.Rev.Lett. 70 (1993) 73 ,Stony Brook preprint ITP-SB-92-57 (1992) and references therein*

[10] P. Schlottmann, *Phys.Rev.B 36 (1987) 5177*
P.A. Bares and G. Blatter, *Phys.Rev.Lett. 64 (1990) 2567*

[11] A. Foerster and M. Karowski, *Nucl.Phys.B 396 (1993) 611*

[12] A.G. Izergin and V.E. Korepin, *Commun.Math.Phys. 79 (1981) 303*

[13] M. Jimbo, *Commun.Math.Phys.102 (1986) 537*
V.V. Bazhanov, *Commun.Math.Phys. 113 (1987) 471*

[14] V.I. Virchirko and N.Yu. Reshetikhin, *Teor.Mat.Fiz. 56 (1983) 260*

[15] V.A. Tarasov, *Theor.Mat.Fiz. 76 (1988) 793*

[16] M. Jimbo, T. Miwa and M. Okado, *Commun.Math.Phys. 116 (1988) 507*
A. Kuniba, *Nucl.Phys.B 355 (1991) 801, and references therein*

[17] M.J. Martins, *" Exact solution of the simplest super-orthosymplectic invariant magnet",preprint UFSCARH-94-12*

[18] M.J. Martins and P.B. Ramos, *J.Phys.A:Math.Gen 27 (1994) 703*

[19] R.B. Zhang, A.J. Bracken and M.D. Gould, *Phys.Lett.B 257 (1991) 133*

[20] Al.B. Zamolodchikov, *Nucl.Phys.B 342 (1990) 746*

[21] F.C. Alcaraz and M.J. Martins ,*J.Phys.A:Math.Gen. 22 (1989) L99*

[22] M.J. Martins, *J.Phys.A: Math.Gen. 24 (1991) L159*

[23] H.J. de Vega and F. Woynarovich, *Nucl.Phys.B 251 (1985) 439*

[24] F. Woynarovich and H-P. Eckle, *J.Phys.A: Math.Gen.20 (1987)L97*

[25] C.J. Hamer, G.R.W. Quispel and M.T. Batchelor, *J.Phys.A: MAth.Gen.20 (1987) 5677*

[26] F. Woynarovich and H-P. Eckle, *J.Phys.A: Math.Gen. 20 (1987) L443*

[27] H. Frahm and V. Korepin, *Phys.Rev.B 42 (1990) 10553; 43 (1991) 5653*

[28] D. Friedan, Z. Qiu and S. Shenker, *Phys.Lett.B 151 (1985) 37*

[29] M.J. Martins, *" Exact solution of the* $Osp(1|2n)$ *spin chain",preprint UFSCARH-94-23*

[30] Yu.G. Stroganov, *Phys.Lett.A 74 (1979) 116*

[31] A. Klümper, *J.Phys.A:Math.Gen. 23 (1990) 809*

[32] J.L. Cardy, *Phase Transitions and Critical Phenomena vol. 11, ed C. Domb and J.L. Lebowitz (New York: Academic) p55*

[33] H. Blöte, J.L. Cardy and M.P. Nightingale, *Phys.Rev.Lett.56 (1986) 742*
I. Affleck, *Phys.Rev.Lett. 56 (1996) 746*

[34] P.P. Kulish and N.Yu. Reshetikhin, *Lett.Math.Phys. 18 (1989) 143*
H. Saleur, *Nucl.Phys.B 336 (1990) 363*

[35] T. Deguchi, A. Fujii and K. Ito, *Phys.Lett.B 238 (1990) 242*

[36] Z. Maassarani, *preprint-USC-94/007, april 1994*

[37] M. Scheunert, W. Nahm and V. Rittenberg, *J.Math.Phys. 18 (1977) 146; 155*

[38] S.O. Warnaar, M.T. Batchelor and B. Nienhuis, *J.Phys.A: Math.Gen. 25 (1992) 3077*

[39] H.J. de Vega and E. Lopes, *Nucl.Phys.B 362 (1991) 261*

Tables

Table 1.(a,b) The complete set of states for the $Osp(1|2)$ Hamiltonian for $L = 4$. The classification is given in terms of the zeros λ_j of the Bethe ansatz equations and their respective energies and momenta. The states with the superscripts $*$ and $\dagger$ indicate double degeneracy ($\lambda_j \to -\lambda_j$) and exact string configuration ,respectively. Some noted special structures have been denoted by A and B.

Table (1.a)

r	$\{n\}$	$E^r(L)/L$	p
0	$1_{3/2}1_{1/2}1_{-1/2}1_{-3/2}$	-1.487116	0
0,2	$1_{-1/2}1_{1/2}$	-0.795334	0
0	$1_{-3/2}1_{-1/2}2_{1/2}$	-0.693713	$\pi/2(3\pi/2)^*$
0	$2_0^\dagger 1_1 1_{-1}$	-0.283594	π
0	A	-0.279860	0
0,2	$1_{3/2}1_{-1/2}$	-0.107625	$\pi/2(3\pi/2)^*$
0,2	$1_{3/2}1_{1/2}$	0	π
0	$2_{1/2}2_{-1/2}$	0.266976	0
0	$1_{-3/2}3_{1/2}$	0.360380	$\pi/2(3\pi/2)^*$
0,2	$2_0^\dagger$	0.5	π
0,2	$1_{3/2}1_{-3/2}$	0.628667	0
0,2	$2_{1/2}$	0.774292	$\pi/2(3\pi/2)^*$
0	$2_0^\dagger B$	0.783594	π
0,2,4	$----$	1.0	0

Table (1.b)

r	$\{n\}$	$E^r(L)/L$	p
1	$1_{-1}1_{0}1_{-1}$	-1.350519	π
1	$1_{0}1_{1}$	-0.567521	$\pi/2(3\pi/2)^*$
1	$1_{0}2_{1}$	$-0.33333\,\dot{3}$	$\pi/2(3\pi/2)^*$
1	$1_{1}1_{-1}$	$-0.16666\,\dot{6}$	0
1	$1_{-1}2_{1}$	0	$\pi/2(3\pi/2)^*$
1,3	1_{0}	0	π
1,3	1_{1}	0.5	$\pi/2(3\pi/2)^*$
1	$2_{0}^{\dagger}$	0.5	π
1	3_{0}	0.638492	π
1	2_{1}	0.734187	$\pi/2(3\pi/2)^*$
1,3	$------$	1.0	0

Table 2. Some complex solutions of the Bethe ansatz equations in the sector $r = 0, 1$.

Table (2)

$\{n\}$	r	p	λ_j
$1_{-3/2}1_{-1/2}2_{1/2}$	0	$\pi/2(3\pi/2)^*$	$-0.681058; -0.1226705; 0.403881 \pm i0.507723$
A	0	0	$\pm i0.097049; \pm i0.936067$
$2_{0}^{\dagger}B$	0	π	$\pm i/2; \pm i1.418833$
$1_{0}2_{1}$	1	$\pi/2(3\pi/2)^*$	$0.106997; 0.696501 \pm i0.464584$
3_{0}	1	π	$0, \pm i1.016413$

Table 3. The imaginary parts of the anomalous solution A for some values of L. $\pm ix$ ($\pm iy$) is the lowest (biggest) imaginary part of A.

Table (3)

L	$\pm ix$	$\pm iy$	$E(L)/L$
4	$\pm i0.097494$	$\pm i0.9360674$	-0.279859
6	$\pm i0.103740$	$\pm i0.908041$	-0.932654
8	$\pm i0.108785$	$\pm i0.894917$	-1.152731
10	$\pm i0.113551$	$\pm i0.887498$	-1.251731
12	$\pm i0.117536$	$\pm i0.882745$	-1.304385

Table 4. Structure of the roots of the ground state for $L = 2$ and $r = 0$ with twisted boundary condition.

Table (4)

ϕ	$\{\lambda_1, \lambda_2\}$	$E(L)/L$
π	$\{0.1160, -0.8216\}$	-1.457
$\frac{5\pi}{4}$	$\{0.0701, -1.0355\}$	-1.339
$\frac{7\pi}{10}$	$\{0.0414, -1.3034\}$	-1.2425
$\frac{3\pi}{2}$	$\{0.0313, -1.4593\}$	-1.2021
$\frac{5\pi}{3}$	$\{0.0124, -2.0702\}$	-1.1088
$\frac{20\pi}{11}$	$\{0.0025, -3.6071\}$	-1.0376
2π	$\{0, \infty\}$	-1

Figures

Figure 1. The 19 nonvanishing Boltzmann weights of the $Osp(1|2)$ chain. Those values not explicitly indicated in the figure are: $a(\lambda,\eta) = -(\lambda-\eta)$; $c(\lambda,\eta) = -\frac{\lambda(\lambda-\eta/2)}{\lambda-3\eta/2}$; $e(\lambda,\eta) = \frac{\lambda\eta}{\lambda-3\eta/2}$; $f(\lambda,\eta) = \eta\frac{2\lambda-3\eta/2}{\lambda-3\eta/2}$ and $g(\lambda,\eta) = \frac{\lambda(\lambda-3\eta/2)-3\eta^2/2}{\lambda-3\eta/2}$.